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2006 J. Phys. A: Math. Gen. 39 371

(http://iopscience.iop.org/0305-4470/39/2/007)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 371–377 doi:10.1088/0305-4470/39/2/007

Algebraic invariants of five qubits

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Received 6 September 2005, in final form 16 November 2005
Published 14 December 2005
Online at stacks.iop.org/JPhysA/39/371

Abstract
We consider the action of the group $SL(2,\mathbb{C})^{\otimes 5}$ on fifth rank tensors on a two-dimensional space, that is, the Hilbert space of a five-qubit system. The Hilbert series of the algebra of polynomial invariants of five qubits pure states is obtained, and the simplest invariants are computed.

PACS numbers: 02.20.Qs, 03.65.Ud, 02.10.Xm

1. Introduction

The invariant theory of *hypermatrices*, which aims to describe the action of the *full product group* $G = SL(V_1) \otimes \cdots \otimes SL(V_r)$ on a tensor space $V_1 \otimes \cdots \otimes V_r$, has recently been connected to various problems in mathematical physics, including calculation of multiple integrals [1, 2], and the investigation of entanglement patterns in quantum information theory. Indeed, quantifying entanglement in multipartite systems is a fundamental issue. However, for systems with more than two parts, very little is known in this respect. A few useful entanglement measures for pure states of three or four qubits have been investigated [3–5], but one is still far from a complete understanding. Furthermore, for systems of up to four qubits, a complete classification of entanglement patterns and of corresponding invariants under the group G, called in this context the group of invertible local filtering operations, is known [6, 7]. Klyachko [8, 9] proposed to associate entanglement (of pure states) in a k-partite system (or perhaps, one should say 'pure k-partite' entanglement) with the mathematical notion of semi-stability, borrowed from geometric invariant theory, which means that at least one G invariant is nonzero. For such states, the absolute values of these invariants provide some kind of entanglement measure. However, even for system of k qubits, the complexity of these invariants grows very rapidly with k. For $k = 2$, they are given by simple linear algebra [10, 11]. The case $k = 3$ is already nontrivial but appears in the physics literature in [12] and boils down to a mathematical result which was known by 1880 [13]. The case $k = 4$ is quite recent [7], and to the best of our knowledge, nothing is known for five-qubit systems[1].

[1] Just after the first version of this note was posted (quant-ph/0506058), A Osterloh and J Siewert informed us of their independent work [14] on the five-qubit problem (see our section 'Conclusion' for a short discussion). Since then, an alternative interpretation of some of our invariants has been given by Lévay [15].

Our main result is a closed expression of the Hilbert series of the algebra of SLOCC invariants of pure five-qubit states. This result, which determines the number of linearly independent homogeneous invariants in any degree, was obtained through intensive symbolic computations relying on a very recent algorithm for multivariate residue calculations [16]. We point out a few properties which can be read off from the series, and determine the simplest invariants, which are of degrees 4 and 6 in the component of the states.

2. Hilbert series

Denote by $V = \mathbb{C}^2$ the local Hilbert space of a two-state particle. The state space of a five-particle system is $\mathcal{H} = V^{\otimes 5}$, which will be regarded as the natural representation of the group of invertible local filtering operations, also known as reversible stochastic local quantum operations assisted by classical communication

$$G = G_{\mathrm{SLOCC}} = \mathrm{SL}(2,\mathbb{C})^{\otimes 5},$$

that is, the group of 5-tuples of complex unimodular 2×2 matrices. We will denote by

$$|\Psi\rangle = \sum_{i_1,i_2,i_3,i_4,i_5=0}^{1} A_{i_1i_2i_3i_4i_5}|i_1\rangle|i_2\rangle|i_3\rangle|i_4\rangle|i_5\rangle$$

a state of the system. An element $\mathbf{g} = \left({}^k g_i^j\right)$ of G maps $|\Psi\rangle$ to the state

$$|\Psi'\rangle = \mathbf{g}|\Psi\rangle$$

whose components are given by

$$A'_{i_1i_2i_3i_4i_5} = \sum_{\mathbf{j}} {}^1g_{i_1}^{j_1}\, {}^2g_{i_2}^{j_2}\, {}^3g_{i_3}^{j_3}\, {}^4g_{i_4}^{j_4}\, {}^5g_{i_5}^{j_5} A_{j_1j_2j_3j_4j_5}. \tag{1}$$

We are interested in the dimension of the space $\mathcal{I}_d$ of all G-invariant homogeneous polynomials of degree $d = 2m$ ($\mathcal{I}_d = 0$ for odd d) in the 32 variables $A_{i_1i_2i_3i_4i_5}$.

It is known that it is equal to the multiplicity of the trivial character of the symmetric group $\mathfrak{S}_{2m}$ in the fifth power of its irreducible character labelled by the partition $[m, m]$, hence given by the following scalar product of characters (cf [17]):

$$\dim \mathcal{I}_d = \langle \chi^{2m} | (\chi^{mm})^5 \rangle = \frac{1}{(2m)!} \sum_{\sigma\in\mathfrak{S}_{2m}} \chi^{mm}(\sigma)^5. \tag{2}$$

The generating function of these numbers

$$h(t) = \sum_{d\geqslant 0} \dim \mathcal{I}_d\, t^d \tag{3}$$

is called the Hilbert series of the algebra $\mathcal{I} = \bigoplus_d \mathcal{I}_d$. Standard manipulations with symmetric functions allow us to express it as a multidimensional residue:

$$h(t) = \oint \frac{\mathrm{d}u_1}{2\pi \mathrm{i} u_1} \cdots \oint \frac{\mathrm{d}u_5}{2\pi \mathrm{i} u_5} \frac{A(\mathbf{u})}{B(\mathbf{u};t)} \tag{4}$$

where the contours are small circles around the origin,

$$A(\mathbf{u}) = \prod_{i=1}^{5} \left(1 + 1/u_i^2\right) \tag{5}$$

and

$$B(\mathbf{u};t) = \prod_{a_i=\pm 1} \left(1 - t\, u_1^{a_1} u_2^{a_2} u_3^{a_3} u_4^{a_4} u_5^{a_5}\right). \tag{6}$$

Table 1. Coefficients of $P(t)$.

n	a_n	n	a_n	n	a_n	n	a_n
0	1	30	24 659	54	225 699	78	9664
8	16	32	36 611	56	214 238	80	5604
10	9	34	52 409	58	195 358	82	3024
12	82	36	71 847	60	172 742	84	1659
14	145	38	95 014	62	146 849	86	770
16	383	40	119 947	64	119 947	88	383
18	770	42	14 849	66	95 014	90	145
20	1659	44	172 742	68	71 847	92	82
22	3024	46	195 358	70	52 409	94	9
24	5604	48	214 238	72	36 611	96	16
26	9664	50	225 699	74	24 659	104	1
28	15 594	52	229 752	76	15 594		

Such multidimensional residues are notoriously difficult to evaluate. After trying various approaches, we eventually succeeded by means of a recent algorithm due to Guoce Xin [16], in a `Maple` implementation. The result can be cast in the form

$$h(t) = \frac{P(t)}{Q(t)} \tag{7}$$

where $P(t)$ is an even polynomial of degree 104 with non-negative integer coefficients a_n,

$$P(t) = \sum_{k=0}^{52} a_{2k} t^{2k}$$

given in table 1 [2], and

$$Q(t) = (1-t^4)^5(1-t^6)(1-t^8)^5(1-t^{10})(1-t^{12})^5.$$

On this expression, it is clear that a complete description of the algebra of G-invariant polynomials by generators and relations is out of reach of any computer system. Nevertheless, inspection of the Hilbert series suggests the following kind of structure for this algebra. We know, since $\dim \mathcal{H} - \dim G = 2^5 - 3 \times 5 = 17$, that there must exist a set of 17 algebraically independent invariants. The denominator of the series, which is precisely a product of 17 factors, makes it plausible that these invariants can be chosen as five polynomials of degree 4 (to be denoted by D_x, D_y, D_z, D_t, D_u), one polynomial of degree 6 (F), five polynomials of degree 8 ($H_1, H_2, \ldots, H_5$), one polynomial of degree 10 (J) and five polynomials of degree 12 ($L_1, \ldots, L_5$). Such a set of 17 polynomials is called a set of *primary invariants*. Their choice is of course not unique.

The numerator should then describe the *secondary invariants*, that is, a set of 3014 400 homogeneous polynomials (1 of degree 0 , 16 of degree 8, 9 of degree 10, 82 of degree 12, etc) such that any invariant polynomial can be uniquely expressed as a *linear* combination of secondary invariants, the coefficients being themselves *arbitrary* polynomials in the primary invariants.

Actually, it is known that the algebra of G-invariants admits such a structure (this is called a *Cohen–Macaulay ring*, see [18]), with a set of 17 primary invariants. But the representation of $h(t)$ as $P(t)/Q(t)$ is not unique, and the knowledge of $h(t)$ is not sufficient to determine

[2] Note that there is no known way to predict this degree without completing the calculation.

the degrees of the primary invariants. However, evidence for our conjecture is provided by the subset of primary invariants computed in section 3.

This picture, which is the simplest kind of description to be expected, is far too complex for physical applications. The best that can be done is to use the Hilbert series as a guide for finding explicitly a small set of reasonably simple invariants, in particular, the primary invariants of lowest degrees. We have computed the first primary invariants, those of degrees 4 and 6, and found good candidates in degree 8, using methods from classical invariant theory (cf [19]).

3. The simplest invariants

3.1. Transvectants and Cayley's Omega process

In order to apply the formalism of classical invariant theory, a state $|\Psi\rangle$ will be interpreted as a quintilinear form on $\mathbb{C}^2$ (called the ground form)

$$f := \sum_{i_1,i_2,i_3,i_4 i_5=0}^{1} A_{i_1 i_2 i_3 i_4 i_5} x_{i_1} y_{i_2} z_{i_3} t_{i_4} u_{i_5}.$$

A covariant of f is a G-invariant polynomial in the coefficients $A_{i_1 i_2 i_3 i_4 i_5}$ and the variables x_i, y_i, z_i, t_i and u_i. A complete set of covariants can, in principle, be computed from the ground form by means of the so-called Omega process (see [19] for notations). Cayley's Omega process consists of applying iteratively differential operators called transvections and defined by

$$(P, Q)^{\epsilon_1\cdots\epsilon_5} = \operatorname{tr} \Omega_x^{\epsilon_1} \cdots \Omega_u^{\epsilon_5} P(x', \ldots, u') Q(x'', \ldots, u'')$$

where

$$\Omega_x = \det \begin{vmatrix} \dfrac{\partial}{\partial x'_0} & \dfrac{\partial}{\partial x'_1} \\ \dfrac{\partial}{\partial x''_0} & \dfrac{\partial}{\partial x''_1} \end{vmatrix}$$

and $\operatorname{tr} : x', x'' \to x$ is the map which erases the primes and the double primes[3].

There is no systematic way to guess which transvectants will lead to interesting (in particular, nonzero) invariants or covariants, but in small degrees, a systematic computer exploration remains possible.

3.2. Degree 4

In degree 4, previous experience of the four-qubit system suggests the following starting point.

Regarding x as a parameter, write f as a quadrilinear binary form in the variables y_i, z_i, t_i and u_i,

$$f = \sum A^x_{i_1 i_2 i_3 i_4} y_{i_1} z_{i_2} t_{i_3} u_{i_4}.$$

It is known that such a quadrilinear form admits an invariant of degree 2 (called Cayley's hyperdeterminant [1, 21, 22]) which is a quadratic binary form

$$b_x := (f, f)^{01\,111} = \alpha x_0^2 + 2\beta x_0 x_1 + \gamma x_1^2 \tag{8}$$

in the variables $x = (x_1, x_2)$. Hence, taking the discriminant $\beta - \alpha\gamma$ of b_x one obtains an invariant D_x of degree 4. We repeat this operation for the other binary variables and obtain four other invariants D_y, D_z, D_t and D_u.

[3] This multivariate version of the Omega process seems to have been first used by Peano in 1882 [20]. His results are reproduced in Olver's book [19].

3.3. Degree 6

We obtain the primary invariant of degree 6 by a succession of transvections. First, we compute a triquadratic covariant of degree 2

$$B_{22\,020} = (f, f)^{00\,101}.$$

This covariant allows us to construct a cubico-quadrilinear covariant of degree 3

$$C_{31\,111} = (B_{22\,020}, f)^{01\,010}$$

which gives a triquadratic polynomial of degree 4

$$D_{22\,200} = (C_{31\,111}, f)^{10\,011}.$$

Hence, one obtains a quintilinear covariant of degree 5

$$E_{11\,111} = (D_{22\,200}, f)^{11\,100}.$$

Finally, we find the invariant of degree 6

$$F = (E_{11\,111}, f)^{11\,111}.$$

3.4. Degree 8

We can compute a set of five linearly independent invariants of degree 8 in the following way. First, we compute some covariants of degree 2 which are triquadratic forms

$$B_{22\,200} = (f, f)^{00\,011}, \qquad B_{00\,222} = (f, f)^{11\,000}.$$

The invariants of degree 4 of these triquadratic forms are invariants of degree 8 of our quintilinear form. Hence, we compute

$$\begin{aligned} D_{40\,000} &= (B_{22\,200}, B_{22\,200})^{02\,200}, & D_{04\,000} &= (B_{22\,200}, B_{22\,200})^{20\,200},\\ D_{00\,400} &= (B_{22\,200}, B_{22\,200})^{22\,000}, & D_{00\,040} &= (B_{00\,222}, B_{00\,222})^{00\,202},\\ D_{00\,004} &= (B_{00\,222}, B_{00\,222})^{00\,220}, \end{aligned}$$

and

$$\begin{aligned} F_x &= (D_{40\,000}, B_{22\,200})^{20\,000}, & F_y &= (D_{04\,000}, B_{22\,200})^{02\,000},\\ F_z &= (D_{00\,400}, B_{22\,200})^{00\,200}, & F_t &= (D_{00\,040}, B_{00\,222})^{00\,020},\\ F_u &= (D_{00\,004}, B_{00\,222})^{00\,002}. \end{aligned}$$

Finally, we find five invariants

$$\begin{aligned} H_x &= (F_x, B_{22\,200})^{22\,200}, & H_y &= (F_y, B_{22\,200})^{22\,200},\\ H_z &= (F_z, B_{22\,200})^{22\,200}, & H_t &= (F_t, B_{00\,222})^{00\,222},\\ H_u &= (F_u, B_{00\,222})^{00\,022}. \end{aligned} \tag{9}$$

3.5. Algebraic independence

To prove that the polynomials $D_x, \ldots, D_u, F, H_x, \ldots, H_u$ are algebraically independent, we need to compute the Jacobian determinant of the set of polynomials $\{D_x, \ldots, D_u, F, H_x, \ldots, H_u, A_{01\,011}, A_{01\,100}, A_{01\,101}, \ldots, A_{11\,111}\}$ for the variables $A_{00\,000}, A_{00\,001}, \ldots, A_{11\,111}$. The direct calculation of this determinant is certainly out of reach on a personal computer, but it is sufficient to compute it for the numerical values given in table 2. This gives the value $-1147\,501\,176\,422\,400 \neq 0$ and implies that the 11 polynomials are algebraically independent. From the Hilbert series the polynomials $D_x, \ldots, D_u, F$ are primary invariants. From the previous computation, we cannot conclude that $H_x, \ldots, H_u$ are primary invariants. Nevertheless, their independence makes them good candidates.

Table 2. Random values of A_{ijklm} used in the Jacobian.

$A_{00\,000}$	$A_{00\,001}$	$A_{00\,010}$	$A_{00\,011}$
1	1	1	3
$A_{00\,100}$	$A_{00\,101}$	$A_{00\,110}$	$A_{00\,111}$
3	2	1	1
$A_{01\,000}$	$A_{01\,001}$	$A_{01\,010}$	$A_{01\,011}$
3	3	3	3
$A_{01\,100}$	$A_{01\,101}$	$A_{01\,110}$	$A_{01\,111}$
1	1	2	1
$A_{10\,000}$	$A_{10\,001}$	$A_{10\,010}$	$A_{10\,011}$
2	2	3	3
$A_{10\,100}$	$A_{10\,101}$	$A_{10\,110}$	$A_{10\,111}$
2	1	3	3
$A_{11\,000}$	$A_{11\,001}$	$A_{11\,010}$	$A_{11\,011}$
2	3	1	1
$A_{11\,100}$	$A_{11\,101}$	$A_{11\,110}$	$A_{11\,111}$
2	3	1	2

Table 3. Evaluation of SLOCC covariants for Osterloh and Siewert states ($\times$ means that the evaluation is not 0).

	$\|\Phi_1\rangle$	$\|\Phi_2\rangle$	$\|\Phi_3\rangle$	$\|\Phi_4\rangle$
D_x	$\times$	$\times$	0	0
D_y	$\times$	$\times$	0	0
D_z	$\times$	0	0	0
D_t	$\times$	0	0	0
D_u	$\times$	0	0	0
F	0	0	0	0
B_x	$\times$	$\times$	$\times$	$\times$
$C_{31\,111}$	0	0	$\times$	$\times$
$E_{11\,111}$	0	$\times$	0	$\times$

4. Conclusion

From the Hilbert series, it appears that the algebra of polynomial invariants of a five-qubit system has a very high complexity. Furthermore, as is already the case with smaller systems [6, 7, 22], the knowledge of the invariants is not sufficient to classify entanglement patterns. In the case of four qubits or three qutrits, this classification can be achieved due to hidden symmetries which have their roots in very subtle aspects of the theory of semi-simple Lie algebras (Vinberg's theory [18]). However, such symmetries are absent in the case of five qubits. Then, the only known general approach for classifying orbits (entanglement patterns) requires the computation of the algebra of covariants, which is already almost intractable in the case of four qubits. It has 170 generators, which have been found [7], but the description of their algebraic relations (syzygies) is definitely out of reach. However, a closer look at the four-qubit system reveals that the classification of Verstraete *et al* [6, 23] can be reproduced by means of only a small set of covariants. We hope that our results will allow the identification and the calculation of such a small set of invariants and covariants, sufficient to separate the physically relevant entanglement patterns, which are probably not so numerous. To illustrate this principle, let us consider a result of Osterloh and Siewert [14]. Having introduced a notion of *filter* which can be used to separate SLOCC orbits in the same way as covariants, these authors show that the four states

$$|\Phi_1\rangle = \frac{1}{\sqrt{2}}(|11\,111\rangle + |00\,000\rangle)$$

$$|\Phi_2\rangle = \frac{1}{2}(|11\,111\rangle + |11\,100\rangle + |00\,010\rangle + |00\,001\rangle)$$

$$|\Phi_3\rangle = \frac{1}{\sqrt{6}}\big(\sqrt{2}|11\,111\rangle + |11\,000\rangle + |00\,100\rangle + |00\,010\rangle + |00\,001\rangle\big)$$

$$|\Phi_4\rangle = \frac{1}{2\sqrt{2}}(\sqrt{3}|11\,111\rangle + |10\,000\rangle + |01\,000\rangle + |00\,100\rangle + |00\,010\rangle + |00\,001\rangle)$$

are in different orbits. As can be seen in table 3, the orbits of these states are also distinguished by our covariants.

Finally, the investigation of entanglement measures requires an understanding of invariants under local unitary transformations (LUT) [24]. The subgroup $K = SU(2)^{\otimes 5}$ has many more invariants than G. We plan to explain in a forthcoming paper how to construct them from a basis of G-covariants.

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